Chapter 6: Cross-Sectional Properties of Structural Members

Introduction
Beam design requires the knowledge of the following.
• Material strengths (allowable stresses)
• Critical shear and moment values
• Cross sectional properties

The shape and proportion of a beam cross section is critical in keeping bending and shear stresses within allowable limits and limiting the deflection that will result from the loads.
1. Why does a 2” x 8” joist standing on edge deflect less when loaded at mid span than the same 2” x 8” used as a plank?
2. Columns with improperly configured cross sections may be highly susceptible to buckling under relatively moderate loads.
3. Are circular pipe columns better at supporting axial loads than columns with a cruciform cross section?

It will be necessary to calculate two cross-sectional properties crucial to the design of beams and columns: the centroid and the moment of inertia.

6.1 Center of Gravity - Centroids
Center of gravity (CG) (or center of mass) refers to masses or weights.
• The center of gravity of a mass or an area is the theoretical point at which the entire mass or area is considered to be concentrated (a.k.a. "balance point").
• If an object were homogeneous, the center of gravity and centroid would coincide.
  - Consider a composite made of Styrofoam and lead.

Center of gravity  

Centroid
The centroid is a geometric property of the volume or area.
- If densities vary, the center of gravity and centroid do not coincide.
- Centroid usually refers to the centers of lines, areas, and volumes.
- The centroid of cross-sectional areas (of beams and columns) will be used later as the reference origin for computing other section properties.

The method of locating the center of gravity is based on the method of determining the resultants of parallel force systems and the principle of moments.
- An area is divided into a number of small areas.
- Each area is represented by a weight acting at its centroid.
- The resultant of the entire area would act through the center of gravity of the total area.

Consider the plate shown at the right.
- The plate is divided into small increments (called components).
- Each component has its own weight and centroid, with all weights directed perpendicular to the surface area (x-y plane).
- The summation of these forces adds up to the total weight of the plate.

\[ \Sigma F_z : \quad W = \Delta W_1 + \Delta W_2 + \Delta W_3 + \cdots + \Delta W_n \]

The centroid is obtained by taking moments about the x and y-axes, respectively.

\[ \Sigma M_y : \quad \bar{x} \ W = \Delta W_1 x_1 + \Delta W_2 x_2 + \Delta W_3 x_3 + \cdots + \Delta W_n x_n \]
\[ \Sigma M_x : \quad \bar{y} \ W = \Delta W_1 y_1 + \Delta W_2 y_2 + \Delta W_3 y_3 + \cdots + \Delta W_n y_n \]

\[ \bar{x} = \frac{\Sigma (x \ \Delta W)}{W} \quad \bar{y} = \frac{\Sigma (y \ \Delta W)}{W} \]
If the plate is divided into an infinite number of elemental pieces, the centroidal expressions may be rewritten in calculus form as follows.

\[ W = \int dW \quad \bar{x} = \int x \, dW/W \quad \bar{y} = \int y \, dW/W \]

Assuming that the plate is of uniform thickness and density, the total weight can be expressed as follows.

\[ W = \gamma t A \]

where

\( W \) = total weight of the plate
\( \gamma \) = density of the plate material
\( t \) = plate thickness
\( A \) = surface area of the plate

Correspondingly, for the component parts (areas) of the plate with uniform thickness, the weight of each part may be expressed as follows.

\[ \Delta W = \gamma t \Delta A \]

where

\( \Delta W \) = weight of component plate area
\( \Delta A \) = surface area of component

If we return to the moment equations written above and substitute the values "\( \gamma t A \)" for "\( W \)" and "\( \gamma t \Delta A \)" for "\( \Delta W \)," we find that, if the plate is homogeneous and of constant thickness, "\( \gamma t \)" cancels out of the equations.

- The resulting moment equations would then be written as follows.
  \[
  \sum M_y : \quad \bar{x} \, A = \Delta A_1 x_1 + \Delta A_2 x_2 + \Delta A_3 x_3 + \cdots + \Delta A_n x_n \\
  \sum M_x : \quad \bar{y} \, A = \Delta A_1 y_1 + \Delta A_2 y_2 + \Delta A_3 y_3 + \cdots + \Delta A_n y_n
  \]

  \[ \bar{x} = \frac{\sum (x \, \Delta A)}{A} \quad \bar{y} = \frac{\sum (y \, \Delta A)}{A} \]

  where

  \[ A = \sum \Delta A \]

The coordinates \( \bar{x} \) and \( \bar{y} \) define the centroid for the area.
The centroids of some of the more common areas have been derived and are shown in Table 6.1 (p. 304) of the text.

To find the centroid of a more complex area (i.e. a composite area), the following procedure may be used.

• The area is first divided into simpler geometric shapes with known centroid locations.

• A reference origin is chosen to establish the reference x- and y-axes.

• The moments of area are summed about the reference x- and y-axes, respectively.

• A tabular solution is often a convenient way to determine the location of the centroid for a composite area.

Symmetry

• When an area (e.g. rectangle, circle, or half-circle) or line possesses an axis of symmetry (i.e. a mirror image on either side of the axis) the centroid of the area must be located on that axis.

• If an area or line possesses two axes of symmetry, the centroid of the area is located at the intersection of the two axes of symmetry.
Example Problem - Centroids

Given: The area shown.

Find: Location of the centroid \((\bar{x}, \bar{y})\).

Solution

<table>
<thead>
<tr>
<th>Part</th>
<th>Area, (A_i)</th>
<th>(\bar{x}_i)</th>
<th>(\bar{y}_i)</th>
<th>(\bar{x}_i A_i)</th>
<th>(\bar{y}_i A_i)</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>10,800</td>
<td>45.0</td>
<td>120.0</td>
<td>486,000</td>
<td>1,296,000</td>
</tr>
<tr>
<td>2</td>
<td>2,700</td>
<td>30.0</td>
<td>40.0</td>
<td>81,000</td>
<td>108,000</td>
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<td>-2,510</td>
<td>73.0</td>
<td>120.0</td>
<td>-183,000</td>
<td>-301,000</td>
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<tr>
<td>Totals</td>
<td>10,990</td>
<td></td>
<td></td>
<td>384,000</td>
<td>1,103,000</td>
</tr>
</tbody>
</table>

\[ A_3 = \pi r^2 / 2 = \pi (40)^2 / 2 = 2,510 \]
\[ x_3 = 90 - 4r/3 \pi = 90 - 4(40)/3\pi = 73.0 \]

\[ \bar{x} = \sum \bar{x}_i A_i / \sum A_i = 384,000/10,990 \]
\[ \bar{x} = 34.9 \text{ mm} \]

\[ \bar{y} = \sum \bar{y}_i A_i / \sum A_i = 1,103,000/10,990 \]
\[ \bar{y} = 100.4 \text{ mm} \]
Given: The cover-plated beam shown.

Find: Neutral axis.

The “neutral axis” is an axis in the cross section of a beam (a member resisting bending) along which there are no longitudinal stresses or strains.
- If the section is symmetric, then the neutral axis is located at the geometric centroid.

A vertical axis through the center of the web forms an axis of symmetry.
- Only the \( \bar{y} \) distance is required.

Use the bottom of the bottom flange as the reference axis.

\[
\bar{y} = \frac{\Sigma y_i A_i}{\Sigma A_i} = \frac{[22.3 \cdot (18.2/2) + 1(12)(18.2 + 1.0/2)]}{[22.3 + 1(12)]} \\
= \frac{(202.93 + 224.40)}{34.3} \\
\bar{y} = 12.46''
\]
6.2 Moment of Inertia of an Area

The moment of inertia (or second-moment as it is sometimes called) is a mathematical expression used in the study of the strength of beams and columns.

\[
I_x = \int y^2 \, dA \\
I_y = \int x^2 \, dA
\]

Moment of inertia (or I-value) is a measure of the effectiveness of the cross-sectional area of a structural element to resist loads.

- Moment of inertia measures a beam’s resistance to bending stress and deflection.
  - A beam section with a large moment of inertia will have smaller stresses and deflections under a given load than one with a lesser moment of inertia.

- Moment of inertia measures the instability or buckling of slender columns.
  - A long, slender column will not be as susceptible to buckling laterally if the moment of inertia of its cross section is sufficient.

Moment of inertia is a measure of cross-sectional stiffness, whereas the modulus of elasticity \(E\) (studied in Chapter 5) is a measure of material stiffness.

Moments of inertia for some basic geometric shapes are shown in Table 6.2 (p. 315) of the text.

For the rectangular cross section, the moments of inertia with respect to the axes passing through the centroid are as follows.

\[
I_x = bh^3/12 \\
I_y = hb^3/12
\]

For the rectangular cross section, the moment of inertia of the area with respect to the base is as follows.

\[
I_x = bh^3/3
\]
For the triangular cross section, the moments of inertia with respect to the axes passing through the centroid are as follows.

\[ I_x = \frac{bh^3}{36} \]
\[ I_y = \frac{hb^3}{36} \]

For the triangular cross section, the moment of inertia of the area with respect to the base is as follows.

\[ I_x = \frac{bh^3}{12} \]

For the circular cross section, the moments of inertia with respect to the axes passing through the centroid are as follows.

\[ I_x = I_y = \frac{\pi d^4}{64} \]
\[ \text{or} \]
\[ I_x = I_y = \frac{\pi r^4}{4} \]

Moment of inertia has the units of length to the fourth power.
- Elements or areas that are relatively far away from the axis will contribute substantially more to an I-value than those that are close to the axis.
6.3  Moment of Inertia of Composite Areas

In steel and concrete construction, the cross-sections usually used for beams and columns are not the simple geometric shapes that are shown in Table 6.2.

- Most structural shapes (e.g. a W-shape) are a composite of two or more simple shapes combined into configurations that produce structural efficiency.
- These shapes are called composite areas.

In structural design, the moment of inertia about the centroidal axis of the cross section is an important section property.

The parallel axis theorem provides a simple way to compute the moment of inertia of a shape about any axis that is parallel to the centroidal axis of the area.

- The principle of the parallel axis theorem may be stated as follows.

> "The moment of inertia of an area with respect to any axis not through its centroid is equal to the moment of inertia of that area with respect to its own parallel centroidal axis plus the product of the area and the square of the distance between the two axes."

For a single area, the parallel axis theorem with respect to the x-axis can be expressed in the following equation form.

\[ I_x = I_{xc} + A \, d_y^2 \]

where

- \( I_x \) = moment of inertia of the area about the x-axis
- \( I_{xc} \) = moment of inertia of the area about its own centroidal x-axis
- \( A \) = area
- \( d_y \) = the perpendicular distance between the x-axis and the parallel axis that passes through the centroid of the area

Similarly, for a single area, the parallel axis theorem with respect to the y-axis can be expressed in the following equation form.

\[ I_y = I_{yc} + A \, d_x^2 \]
For composite areas, the parallel axis theorem with respect to the x-axis can be expressed as follows.

\[ I_x = I_{xc1} + A_1 (dy_1)^2 + I_{xc2} + A_2 (dy_2)^2 + \cdots \]
\[ = [I_{xc1} + I_{xc2} + \cdots ] + [A_1 (dy_1)^2 + A_2 (dy_2)^2 + \cdots ] \]
\[ I_x = \sum I_{xc} + \sum A \, dy^2 \]

Similarly, for composite areas, the parallel axis theorem with respect to the y-axis can be expressed as follows.

\[ I_y = I_{yc1} + A_1 (dx_1)^2 + I_{yc2} + A_2 (dx_2)^2 + \cdots \]
\[ = [I_{yc1} + I_{yc2} + \cdots ] + [A_1 (dx_1)^2 + A_2 (dx_2)^2 + \cdots ] \]
\[ I_y = \sum I_{yc} + \sum A \, dx^2 \]

A tabular solution is often a convenient way to determine the moment of inertia for a composite area.
Example Problems - Moment of Inertia of Composite Areas

Given: The composite area shown.

Find: \( I_x \) and \( I_y \)

<table>
<thead>
<tr>
<th>Part</th>
<th>( A_i )</th>
<th>( \bar{x}_i )</th>
<th>( \bar{y}_i )</th>
<th>( A_i \bar{x}_i )</th>
<th>( A_i \bar{y}_i )</th>
<th>( A_i \bar{x}_i^2 )</th>
<th>( A_i \bar{y}_i^2 )</th>
<th>( I_{xi} )</th>
<th>( I_{yi} )</th>
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<td>1.50</td>
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<td>-4.54</td>
<td>-1.74</td>
<td>-6.28</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- \( \bar{x}_3 = 4R/3\pi = 4(6)/3\pi = 2.55 = \bar{y}_3 \)
- \( \bar{y}_4 = -4R/3\pi = -4(2)/3\pi = -0.85 \)
- \( \bar{I}_{x1} = bh^3/12 = 3(9)^3/12 = 182.25 \)
- \( \bar{I}_{x2} = bh^3/36 = 6(9)^3/36 = 121.50 \)
- \( \bar{I}_{x3} = 0.0549R^4 = 0.0549(6)^4 = 71.15 \)
- \( \bar{I}_{x4} = 0.109R^4 = 0.109(2)^4 = 1.74 \)
- \( \bar{I}_{y1} = bh^3/12 = 9(3)^3/12 = 20.25 \)
- \( \bar{I}_{y2} = bh^3/36 = 9(6)^3/36 = 54.00 \)
- \( \bar{I}_{y3} = \bar{I}_{x3} = 71.15 \)
- \( \bar{I}_{y4} = (\pi/8)R^4 = (\pi/8)(2)^4 = 6.28 \)

Use \( \bar{I}_{x4} = -1.74 \) (negative area)

- The moments of inertia with respect to the \( x \)- and \( y \)-axes are determined as follows.
  - \( I_x = \sum \bar{I}_{xi} + \sum \bar{y}_i^2A_i = 373.16 + 240.04 = 613.20 \text{ in}^4 \)
  - \( I_y = \sum \bar{I}_{yi} + \sum \bar{x}_i^2A_i = 139.12 + 863.06 = 1002.18 \text{ in}^4 \)

- The location of the centroid is determined as follows.
  - \( \bar{x} = \sum \bar{x}_iA_i/\sum A_i = -84.57/75.99 = -1.11" \)
  - \( \bar{y} = \sum \bar{y}_iA_i/\sum A_i = 117.93/75.99 = 1.55" \)

- The centroidal moments of inertia for the composite area are determined as follows.
  - \( \bar{I}_{x} = I_x - \bar{y}^2A = 613.20 - (1.55)^2 \times 75.99 = 430.63 \text{ in}^4 \)
  - \( \bar{I}_{y} = I_y - \bar{x}^2A = 1002.18 - (-1.11)^2 \times 75.99 = 908.55 \text{ in}^4 \)

6.11
Given: The cover-plated beam shown.

Find: The moment of inertia with respect to a horizontal axis through the centroid of the area (i.e. the neutral axis).

A vertical axis through the center of the web forms an axis of symmetry.

- Only the \( \bar{y} \) distance is required.

Find the location of the centroid (which corresponds with the neutral axis).

- Use the bottom of the bottom flange as the reference axis.

\[
\bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i} = \frac{31.7 (29.8/2) + 1(18)(29.8 + 1.0/2)}{31.7 + 1(18)}
\]

\[
= \frac{472.33 + 545.40}{49.7} = 20.48\text{”}
\]

Using the “Parallel Axis Theorem”, determine the moment of inertia with respect to a horizontal axis through the centroid of the area (i.e. the neutral axis).

\[
I_x = (I_x)_\text{beam} + (I_x)_\text{plate}
\]

\[
= [4470 + 31.7(29.8/2 - 20.48)^2] + [18(1)^3/12 + 1(18)(29.8 + 1.0/2 - 20.48)^2]
\]

\[
= (4470 + 987.02) + (1.50 + 1735.78)
\]

\[
I_x = 7194.30 \text{ in}^4
\]
6.4 Radius of Gyration

In the study of columns (Chapter 9) we will be using a section property known as the **radius of gyration**.

- The *radius of gyration* (designated as "$r_x$" or "$r_y$") expresses the relationship between the area of a cross section and the moment of inertia.
- The radius of gyration is a shape factor that measures a column's resistance to buckling about an axis.
- The larger the $r$-value, the more resistance there is to buckling.

Consider an area $A$ that has a moment of inertia $I_x$ with respect to the x-axis.

- Let the area is concentrated into a thin strip parallel to the x-axis.
- The area has the same moment of inertia as the original area.
- The strip is placed at a distance $r_x$ from the axis.

Then \[ I_x = A \, r_x^2 \]
\[ r_x^2 = \frac{I_x}{A} \]
and \[ r_x = \left(\frac{I_x}{A}\right)^{1/2} \]

Similarly, \[ I_y = A \, r_y^2 \]
\[ r_y^2 = \frac{I_y}{A} \]
and \[ r_y = \left(\frac{I_y}{A}\right)^{1/2} \]